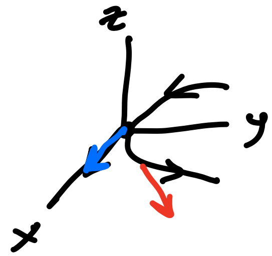
Closing Thu: *13.2*
Closing ***Sunday****:* *13.3*

Exam 1 is Tues in quiz section

- covers 12.1-12.6, 13.1-13.3
- 4 pages of questions
- You get 50 minutes
- Allowed
  - One 8.5 by 11 inch sheet of handwritten notes (both sides)
  - Ti-30x IIS calculator

Here are some of my resources:

- Exam 1 Review
- Exam 1 Fact Sheet
- My Exam Archive
- Review HW and work thru ***several*** old exams to prepare.



13.3 Measurement Tools for Curves

Warm Up: *Consider* $\boldsymbol{r}(t) = \langle t, t^2, 0\rangle$ z=0 xy-plane

(a) Are $\boldsymbol{r}'(t)$ and $\boldsymbol{r}''(t)$ always orthogonal?

(b) Find $\boldsymbol{T}(t)$.

$r'(t) = \langle 1, 2t, 0\rangle$ Tangent

$r''(t) = \langle 0, 2, 0\rangle$ acceleration

$r'(t) \cdot r''(t) = 0 + 4t + 0 \overset{?}{=} 0$

$\boxed{\text{No, only when } t=0}$

$T(t)$ = unit tangent

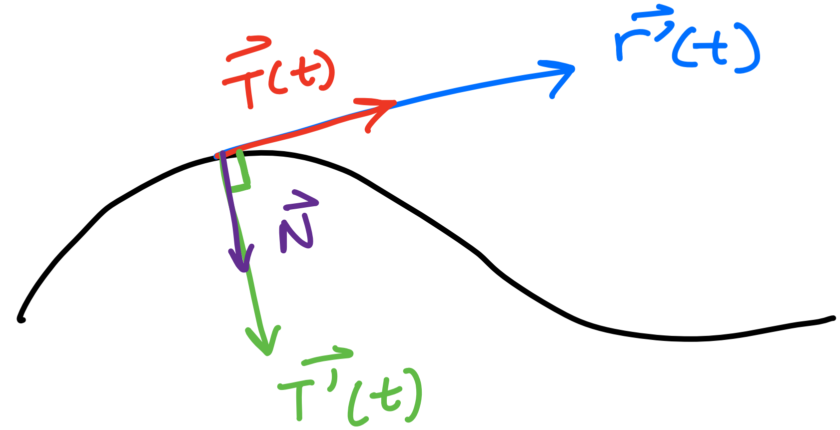
$= \frac{1}{\sqrt{1+4t^2}}\langle 1, 2t, 0\rangle$

$T(t) = \boxed{\left\langle \frac{1}{\sqrt{1+4t^2}}, \frac{2t}{\sqrt{1+4t^2}}, 0\right\rangle}$

$\vec{T}'(t)$ is always perpendicular to $\vec{T}(t)$

Important Conceptual Fact

Thm: $\boldsymbol{T}$ and $\boldsymbol{T}'$ are always orthogonal.



Proof:

Since $\boldsymbol{T} \cdot \boldsymbol{T} = |\boldsymbol{T}|^2 = 1$, constant

we differentiate both sides to get

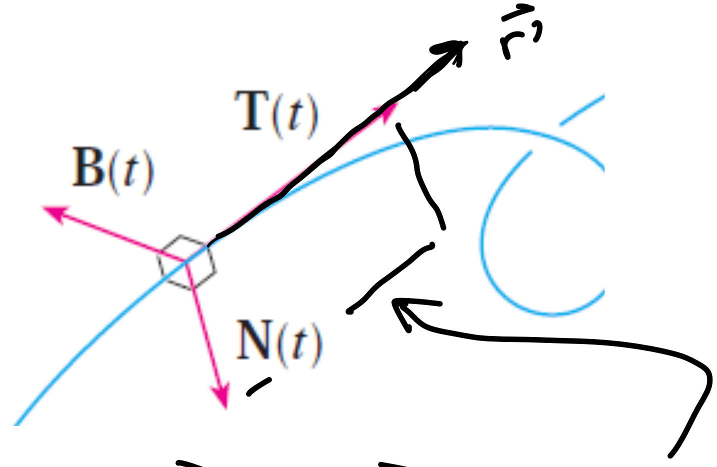
$$\boldsymbol{T}' \cdot \boldsymbol{T} + \boldsymbol{T} \cdot \boldsymbol{T}' = 0.$$

So $2\boldsymbol{T} \cdot \boldsymbol{T}' = 0.$

Thus, $\boldsymbol{T} \cdot \boldsymbol{T}' = 0.$ (QED)

1. $\vec{T}$
2. $\vec{T}'$ ← points "inward"

Part 1: TNB-Frame



$\vec{T}(t)$, $\vec{N}(t)$ $\vec{r}'$ and $\vec{r}''$ are all on the same plane

$$\boldsymbol{T}(t) = \frac{\boldsymbol{r}'(t)}{|\boldsymbol{r}'(t)|} = \text{unit tangent}$$

$$\boldsymbol{N}(t) = \frac{\boldsymbol{T}'(t)}{|\boldsymbol{T}'(t)|} = \text{principal unit normal}$$

$$\cancel{\boldsymbol{B}(t) = \boldsymbol{T}(t) \times \boldsymbol{N}(t) = \text{binormal}}$$

Not in the class anymore

TNB-Frame Cool Facts:

- $\boldsymbol{T}(t)$ and $\boldsymbol{N}(t)$ together give a good approximation of the "plane of motion" called the *osculating (kissing) plane.*

- $\boldsymbol{T}(t)$, $\boldsymbol{N}(t)$, $\boldsymbol{r}'(t)$, and $\boldsymbol{r}''(t)$ are ALL parallel to the osculating plane

Good news: Useful measurement tools.
Bad news: TNB vectors have messy formulas.

Basic Visual: https://www.math3d.org/f1rRKiN7

Sp’18 – Exam 1 – Loveless

Find the principal unit normal vector $\underline{\mathbf{N}}(t)$ for
$\boldsymbol{r}(t) = < 3t, \cos(4t), \sin(4t) >$.

$\sin^2 + \cos^2 = 1$

$\vec{r}'(t) = < 3, -4\sin(4t), 4\cos(4t) >$

$|\vec{r}'| = \sqrt{9 + 16\sin^2(4t) + 16\cos^2(4t)} = \sqrt{9+16} = \sqrt{25} = 5$

$\vec{T}(t) = < \frac{3}{5}, -\frac{4}{5}\sin(4t), \frac{4}{5}\cos(4t) >$

$\vec{T}'(t) = < 0, -\frac{16}{5}\cos(4t), -\frac{16}{5}\sin(4t) >$

$|\vec{T}'| = \sqrt{0 + \frac{16}{5}^2\cos^2(4t) + \frac{16}{5}^2\sin^2(4t)} = \sqrt{\left(\frac{16}{5}\right)^2} = \frac{16}{5}$

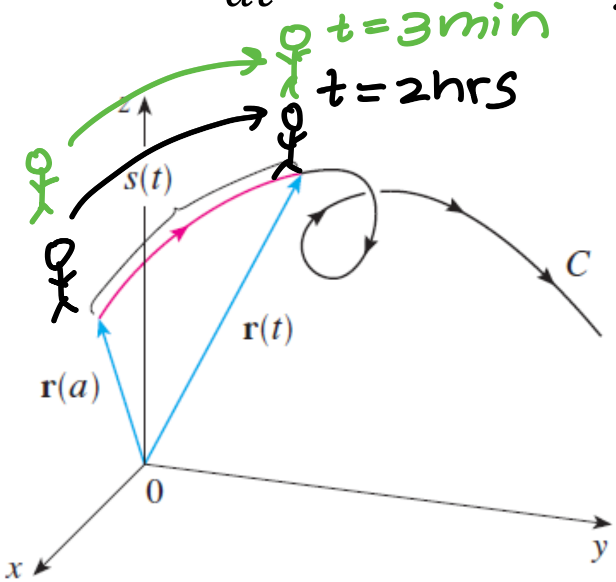
$\vec{N}(t) = \frac{T'(t)}{|T'(t)|} = < 0, -\cos(4t), -\sin(4t) >$

Visuals: https://www.math3d.org/eiaTISrA

Part 2: Arc Length (*i.e.* Distance)

$$s(t) = \int_a^t \sqrt{(x'(u))^2 + (y'(u))^2 + (z'(u))^2}\, du = \int_a^t \underbrace{|\vec{r}'(u)|}_{\text{speed}}\, du = \text{distance}$$

$$\frac{ds}{dt} = |\vec{r}'(t)| = \text{speed}$$



$t=0$ $\quad t=1$

$t^3=0 \quad t^3=1$

$3t^2=0 \quad 3t^2=3$

$6t=0 \quad 6t=6$

Sp'19 – Exam 1 – Loveless

Find the distance along the curve $\boldsymbol{r}(t) = \langle t^3, 3t^2, 6t\rangle$ from (0,0,0) to (1,3,6).

$$\int_0^1 \sqrt{(3t^2)^2 + (6t)^2 + 6^2}\, dt$$

$$\int_0^1 \sqrt{9t^4 + 36t^2 + 36}\, dt$$

$$\int_0^1 3\sqrt{t^4 + 4t^2 + 4}\, dt$$

$$\int_0^1 3(t^2+2)\, dt = \int_0^1 3t^2 + 6\, dt$$

$$= t^3 + 6t \Big|_0^1 = \boxed{7}$$

Visual: https://www.math3d.org/U6VxUFDG

Reparameterizing a curve in terms of distance traveled:

$x = 1 + 3u$
$y = 2u$
$z = 5u$

$\boxed{u = \text{TIME}}$

speed = $\frac{\text{distance}}{\text{time}}$

Example

1. Find the length of the curve

dist.

$$\boldsymbol{r}(u) = \langle 1 + 3\text{u}, 2\text{u}, 5u \rangle$$

from 0 to t.

speed $= \sqrt{38}$ mph

$$s = \int_0^t \underbrace{\sqrt{3^2 + 2^2 + 5^2}}_{\text{speed}} \, du$$

$$= \int_0^t \sqrt{38} \, dt$$

$$\sqrt{38}\, u \Big|_0^t = \sqrt{38}\, t$$

$$\boxed{s = \sqrt{38}\, t}$$

distance $= \sqrt{38}$ (Time)

2. Reparameterize the curve in terms of arc length, *s*.

$x = 1 + 3(\text{Time}) = 1 + 3t$

$y = 2(\text{Time}) = 2t$

$z = 5(\text{Time}) = 5t$

OR

$x = 1 + 3 \frac{\text{dist}}{\sqrt{38}} = 1 + \frac{3s}{\sqrt{38}}$

$y = 2 \frac{\text{dist}}{\sqrt{38}} = \frac{2s}{\sqrt{38}}$

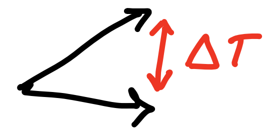
$z = 5 \frac{\text{dist}}{\sqrt{38}} = \frac{5s}{\sqrt{38}}$

Curvature

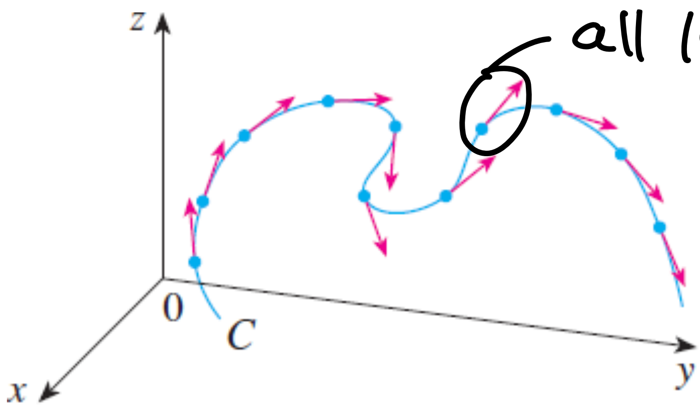
The **curvature** at a point, K, is a measure of how quickly a curve is changing direction at that point.

$$K = \frac{change\ in\ direction}{change\ in\ distance} = \left|\frac{d\vec{\boldsymbol{T}}}{ds}\right| = \frac{\Delta T}{\Delta S}$$

messy!



all length 1



Roughly, how much does your direction change if you move a small amount (“one inch”) along the curve?

$$K \approx \left|\frac{\vec{\boldsymbol{T}}_2 - \vec{\boldsymbol{T}}_1}{\text{"one inch"}}\right| = \left|\frac{\Delta\vec{\boldsymbol{T}}}{\Delta s}\right|$$

This is NOT easy to compute directly from arc length, so instead we use this…

$$K(t) = \left|\frac{d\vec{\boldsymbol{T}}}{ds}\right| = \frac{|\vec{\boldsymbol{T}}'(t)|}{|\vec{\boldsymbol{r}}'(t)|} = \frac{|\vec{\boldsymbol{r}}'(t) \times \vec{\boldsymbol{r}}''(t)|}{|\vec{\boldsymbol{r}}'(t)|^3}$$

Cool Side Fact: The *radius of curvature* is the radius of the circle that would best fit this curve. It is always 1/K.

in 2D

$\langle x, f(x), 0 \rangle$

Plug into here

In 2D this becomes:

$$K(x) = \frac{|\boldsymbol{r}' \times \boldsymbol{r}''|}{|\boldsymbol{r}'|^3} = \frac{|f''(x)|}{\left(1 + \left(f'(x)\right)^2\right)^{3/2}}$$

Proof of shortcut (feel free to ignore):

Theorem: $\frac{|\boldsymbol{T}'(t)|}{|\boldsymbol{r}'(t)|} = \frac{|\boldsymbol{r}'(t)\times\boldsymbol{r}''(t)|}{|\boldsymbol{r}'(t)|^3}$

Proof:

Since $\boldsymbol{T}(t) = \frac{\boldsymbol{r}'(t)}{|\boldsymbol{r}'(t)|}$, we have

$$\boldsymbol{r}'(t) = |\boldsymbol{r}'(t)|\boldsymbol{T}(t).$$

Differentiating this gives (prod. rule):

$$\boldsymbol{r}''(t) = |\boldsymbol{r}'(t)|'\boldsymbol{T}(t) + |\boldsymbol{r}'(t)|\boldsymbol{T}'(t).$$

Take cross-prod. of both sides with $\vec{\boldsymbol{T}}$:

$$\boldsymbol{T}\times\boldsymbol{r}'' = |\boldsymbol{r}'|'\,(\boldsymbol{T}\times\boldsymbol{T}) + |\boldsymbol{r}'|\,(\boldsymbol{T}\times\boldsymbol{T}').$$

Since $\boldsymbol{T}\times\boldsymbol{T} = <0,0,0>$ (why?)
and $\boldsymbol{T} = \frac{\boldsymbol{r}'}{|\boldsymbol{r}'|}$, we have

$$\frac{\boldsymbol{r}'\times\boldsymbol{r}''}{|\boldsymbol{r}'|} = |\boldsymbol{r}'|\,(\boldsymbol{T}\times\boldsymbol{T}').$$

Taking the magnitude gives (why?)

$$\frac{|\boldsymbol{r}'\times\boldsymbol{r}''|}{|\boldsymbol{r}'|} = |\boldsymbol{r}'|\,|\boldsymbol{T}\times\boldsymbol{T}'| = |\boldsymbol{r}'|\,|\boldsymbol{T}||\boldsymbol{T}'|sin\left(\frac{\boldsymbol{\pi}}{\mathbf{2}}\right),$$

Since $|\boldsymbol{T}| = 1$, we have

$$|\boldsymbol{T}'| = \frac{|\boldsymbol{r}'\times\boldsymbol{r}''|}{|\boldsymbol{r}'|^2}$$

Therefore

$$K = \left|\frac{d\boldsymbol{T}}{ds}\right| = \frac{|\boldsymbol{T}'(t)|}{|\boldsymbol{r}'(t)|} = \frac{|\boldsymbol{r}'\times\boldsymbol{r}''|}{|\boldsymbol{r}'|^3}.$$

Proof for 2D (again feel free to ignore)

To find curvature for a 2D function, *y = f(x)*, we can form a 3D vector function as follows

$$\boldsymbol{r}(x) = \langle x, f(x), 0\rangle$$

so $\boldsymbol{r}'(x) = \langle 1, f'(x), 0\rangle$ and

$$\boldsymbol{r}''(x) = \langle 0, f''(x), 0\rangle$$

Taking the cross-product gives…

$$\boldsymbol{r}' \times \boldsymbol{r}'' = \langle 0,0, f''(x)\rangle$$

Note that

$$|\boldsymbol{r}' \times \boldsymbol{r}''| = \sqrt{0^2 + 0^2 + \left(f''(x)\right)^2} = |f''(x)|$$

and

$$|\boldsymbol{r}'(x)| = \sqrt{1 + (f'(x))^2}$$

Thus,

$$K(x) = \frac{|\boldsymbol{r}' \times \boldsymbol{r}''|}{|\boldsymbol{r}'|^3} = \frac{|f''(x)|}{\left(1 + \left(f'(x)\right)^2\right)^{3/2}}$$

Sp'19 – Exam 1 – Loveless

Find the curvature of

$\boldsymbol{r}(t) = \langle t^3, 3t^2, 6t \rangle$ at t = 0.

$r'(t) = < 3t^2, 6t, 6 >$

$r''(t) = < 6t, 6, 0 >$

$$r'(0) \times r''(0) = \begin{vmatrix} i & j & k \\ 0 & 0 & 6 \\ 0 & 6 & 0 \end{vmatrix} = (0-36)i - (0)j + (0)k$$

$$= < -36, 0, 0 >$$

$$\sqrt{r'(0) \times r''(0)} = \sqrt{36^2} = 36$$

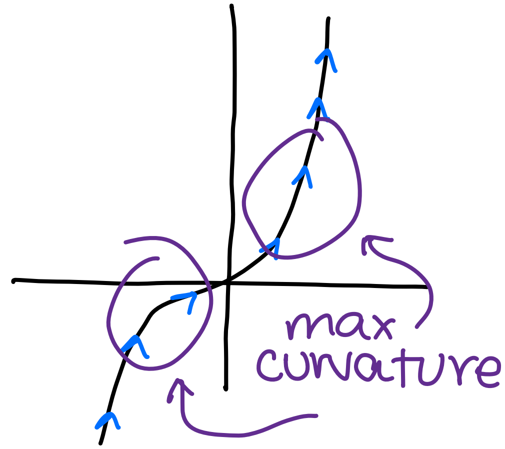
$$\sqrt{r'(0)} = \sqrt{6^2} = 6$$

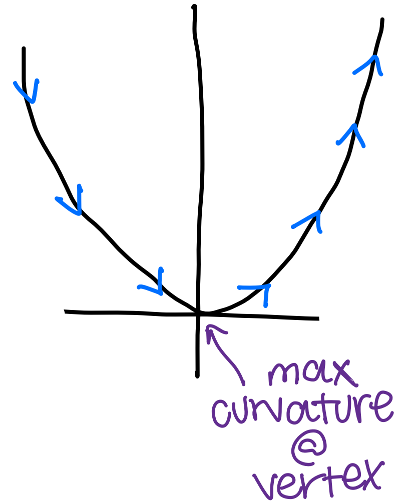
$$k(t) = \frac{36}{6^3} = \boxed{\frac{1}{6}}$$

Visuals: https://www.math3d.org/ExiDP27d

F'18 – Exam 1 – Loveless

For x > 0, find the x-value at which curvature is maximum for $f(x) = \frac{x^3}{3}$.





$f(x) = \frac{1}{3}x^3$ = height

$f'(x) = x^2$ = slope

$f''(x) = 2x$ = concavity

$$K(x) = \frac{|f''(x)|}{\left(1+(f'(x))^2\right)^{3/2}} = \frac{|2x|}{\left(1+(x^2)^2\right)^{3/2}}$$

$$y = \frac{2x}{(1+x^4)^{3/2}} \qquad \frac{dy}{dx} = \frac{(1+x^4)^{3/2}\cdot 2 - 2x\cdot\frac{3}{2}(1+x^4)^{1/2}\cdot 4x^3}{(1+x^4)^3} = 0$$

$$2(1+x^4)^{3/2} - 12x^4(1-x^4)^{1/2} = 0$$

$$2(1+x^4)^{1/2}\left[(1+x^4) - 6x^4\right] = 0$$

$$1-5x^4 = 0 \qquad x^4 = \frac{1}{5} \qquad x = \sqrt[4]{1/5}$$

$$5x^4 = 1$$

Visuals: https://www.desmos.com/calculator/uutovrkvos

Preview of 13.4: Acceleration

If ***t = time*** and position is given by

$$\boldsymbol{r}(t) = \langle x(t), y(t), z(t)\rangle$$

then

$$\boldsymbol{r}'(t) = \lim_{h\to 0}\frac{\boldsymbol{r}(t+h) - \boldsymbol{r}(t)}{h} = \text{velocity}$$

$$|\boldsymbol{r}'(\boldsymbol{t})| = \frac{\text{change in dist}}{\text{change in time}} = \text{speed} = \frac{\text{ds}}{\text{dt}}$$

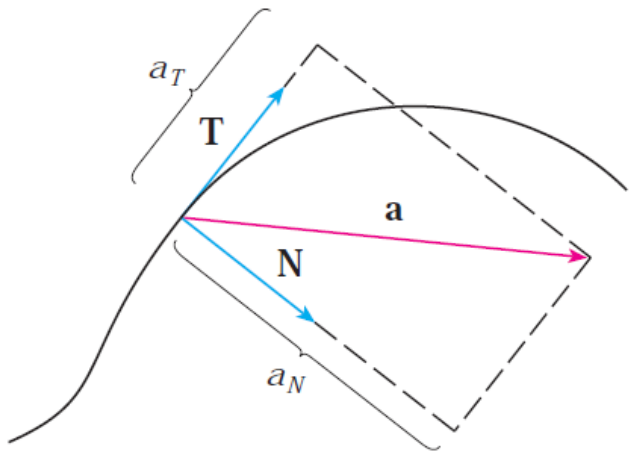
$$\boldsymbol{r}''(t) = \lim_{h\to 0}\frac{\boldsymbol{r}'(t+h) - \boldsymbol{r}'(t)}{h}$$

$$= \frac{\text{change in velocity}}{\text{change in time}} = \boldsymbol{a}(t)$$

Since $Force = \boldsymbol{F} = m \cdot \boldsymbol{a}$, we can think of acceleration as the way the object is being "pulled".

Sometimes acceleration causes the object to speed up or slow down and sometimes it makes the object turn.

Tangential and Normal Components of Acceleration



Definition:

$a_T = \text{comp}_{\boldsymbol{T}}(\boldsymbol{a}) = \boldsymbol{a} \cdot \boldsymbol{T}$ = tangential comp.

$a_N = \text{comp}_{\boldsymbol{N}}(\boldsymbol{a}) = \boldsymbol{a} \cdot \boldsymbol{N}$ = normal comp.

which can be rewritten as…

$$a_T = \frac{\boldsymbol{r}' \cdot \boldsymbol{r}''}{|\boldsymbol{r}'|} \text{ and } a_N = \frac{|\boldsymbol{r}' \times \boldsymbol{r}''|}{|\boldsymbol{r}'|}$$

Notes:

- a_T can be positive or negative (or zero)
  positive – speedometer speed increasing
  negative – speedometer speed decreasing

- a_N is always positive (or zero)
  (accel. points "inward" relative to the curve, but not always "directly" inward)

For interpreting use,

$a_T = v' = \frac{d}{dt}|r'(t)|$ = "deriv. of speed"

$a_N = kv^2$ = curvature · (speed)^2

Projectile Visual: https://www.math3d.org/QbuedSnK